The hyperbolic version of the Askey-Wilson difference equation involves two positive scale parameters $a_+$ and $a_-$, and four real coupling parameters $c_0, c_1, c_2, c_3$. We show that this equation admits a basis of two elementary meromorphic solutions for a dense set in the parameter space $(0, \infty)^2 \times \mathbb{R}^4$. The solutions are obtained via sixteen parameter shifts associated with the weight lattice of the Lie algebra $D_4$. We compare these solutions to previously known ones, observing they are minimal (in a sense coming from Nevanlinna theory) in the space of all meromorphic solutions.